CALABI-YAU HYPERSURFACES IN THE DIRECT PRODUCT OF $\mathbb{P}^1$ AND INERTIA GROUPS

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Abstract. We produce the family of Calabi-Yau hypersurfaces X_n of $(\mathbb{P}^1)^{n+1}$ in higher dimension whose inertia group contains non commutative free groups. This is completely different from Takahashi's result [Ta98] for Calabi-Yau hypersurfaces M_n of $\mathbb{P}^{n+1}$.

1. Introduction

Throughout this paper, we work over $\mathbb{C}$. Given an algebraic variety X, it is natural to consider its birational automorphisms $\varphi\colon X \dashrightarrow X$. The set of these birational automorphisms forms a group $\mathrm{Bir}(X)$ with respect to the composition. When X is a projective space $\mathbb{P}^n$ or equivalently an n-dimensional rational variety, this group is called the Cremona group. In higher dimensional case ($n \geq 3$), though many elements of the Cremona group have been described, its whole structure is little known.

Let V be an $(n+1)$-dimensional smooth projective rational manifold. In this paper, we treat subgroups called the "inertia group" (defined below (1.1)) of some hypersurface $X \subset V$ originated in [Gi94]. It consists of those elements of the Cremona group that act on X as identity.

In Section 3, we mention the result (Theorem 3.2) of Takahashi [Ta98] about the smooth Calabi-Yau hypersurfaces M_n of $\mathbb{P}^{n+1}$ of degree $n+2$ (that is, M_n is a hypersurface such that it is simply connected, there is no holomorphic k-form on M_n for $0 < k < n$, and there is a nowhere vanishing holomorphic n-form ω_{M_n}). It turns out that the inertia group of M_n is trivial (Theorem 1.4). Takahashi's result (Theorem 3.2) is proved by using the "Noether-Fano inequality". It is the useful result that tells us when two Mori fiber spaces are isomorphic. Theorem 1.4 is a direct consequence of Takahashi's result.

In Section 4, we consider Calabi-Yau hypersurfaces

$$X_n = (2, 2, \dots, 2) \subset (\mathbb{P}^1)^{n+1}.$$

Let

$$\mathrm{UC}(N) := \overbrace{\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \cdots * \mathbb{Z}/2\mathbb{Z}}^{N} = \mathop{\ast}_{i=1}^{N} \langle t_i \rangle$$

be the *universal Coxeter group* of rank N where $\mathbb{Z}/2\mathbb{Z}$ is the cyclic group of order 2. There is no non-trivial relation between its N natural generators t_i. Let

$$p_i\colon X_n \to (\mathbb{P}^1)^n \quad (i = 1, \dots, n+1)$$

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be the natural projections which are obtained by forgetting the i-th factor of $(\mathbb{P}^1)^{n+1}$. Then, the $n+1$ projections p_i are generically finite morphism of degree 2. Thus, for each index i, there is a birational transformation

$$\iota_i\colon X_n \dashrightarrow X_n$$

that permutes the two points of general fibers of p_i and this provides a group homomorphism

$$\Phi\colon \mathrm{UC}(n+1) \to \mathrm{Bir}(X_n).$$

From now, we set $P(n+1) \coloneqq (\mathbb{P}^1)^{n+1}$. Cantat-Oguiso proved the following theorem in [CO11].

Theorem 1.1. ([CO11, Theorem 1.3 (2)]) *Let X_n be a generic hypersurface of multidegree $(2, 2, \dots, 2)$ in $P(n+1)$ with $n \geq 3$. Then the morphism Φ that maps each generator t_j of $\mathrm{UC}(n+1)$ to the involution ι_j of X_n is an isomorphism from $\mathrm{UC}(n+1)$ to $\mathrm{Bir}(X_n)$.*

Here "generic" means X_n belongs to the complement of some countable union of proper closed subvarieties of the complete linear system $\left|(2, 2, \dots, 2)\right|$.

Let $X \subset V$ be a projective variety. The *decomposition group* of X is the group

$$\mathrm{Dec}(V, X) \coloneqq \{f \in \mathrm{Bir}(V) \mid f(X) = X \text{ and } f|_X \in \mathrm{Bir}(X)\}.$$

The *inertia group* of X is the group

$$\mathrm{Ine}(V, X) \coloneqq \{f \in \mathrm{Dec}(V, X) \mid f|_X = \mathrm{id}_X\}. \tag{1.1}$$

Then it is natural to consider the following question:

Question 1.2. Is the sequence

$$1 \longrightarrow \mathrm{Ine}(V, X) \longrightarrow \mathrm{Dec}(V, X) \xrightarrow{\gamma} \mathrm{Bir}(X) \longrightarrow 1 \tag{1.2}$$

exact, i.e., is γ surjective?

Note that, in general, this sequence is not exact, i.e., γ is not surjective (see Remark 1.5). When the sequence (1.2) is exact, the group $\mathrm{Ine}(V, X)$ measures how many ways one can extend $\mathrm{Bir}(X)$ to the birational automorphisms of the ambient space V.

Our main result is following theorem, answering a question asked by Ludmil Katzarkov:

Theorem 1.3. *Let $X_n \subset P(n+1)$ be a smooth hypersurface of multidegree $(2, 2, \dots, 2)$ and $n \geq 3$. Then:*

(1) *$\gamma\colon \mathrm{Dec}(P(n+1), X_n) \to \mathrm{Bir}(X_n)$ is surjective, in particular Question 1.2 is affirmative for X_n.*
(2) *If, in addition, X_n is generic, there are $n+1$ elements ρ_i $(1 \leq i \leq n+1)$ of $\mathrm{Ine}(P(n+1), X_n)$ such that*

$$\langle \rho_1, \rho_2, \dots, \rho_{n+1}\rangle \simeq \underbrace{\mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}}_{n+1} \subset \mathrm{Ine}(P(n+1), X_n).$$

In particular, $\mathrm{Ine}(P(n+1), X_n)$ is an infinite non-commutative group.

Our proof of Theorem 1.3 is based on an explicit computation of elementary flavour.

We also consider another type of Calabi-Yau manifolds, namely smooth hypersurfaces of degree $n+2$ in $\mathbb{P}^{n+1}$ and obtain the following result:

Theorem 1.4. *Suppose $n \geq 3$. Let $M_n = (n+2) \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $n+2$. Then Question 1.2 is also affirmative for M_n. More precisely:*

(1) $\mathrm{Dec}(\mathbb{P}^{n+1}, M_n) = \{f \in \mathrm{PGL}(n+2, \mathbb{C}) = \mathrm{Aut}(\mathbb{P}^{n+1}) \mid f(M_n) = M_n\}$.

(2) $\mathrm{Ine}(\mathbb{P}^{n+1}, M_n) = \{\mathrm{id}_{\mathbb{P}^{n+1}}\}$, *and* $\gamma \colon \mathrm{Dec}(\mathbb{P}^{n+1}, M_n) \xrightarrow{\simeq} \mathrm{Bir}(M_n) = \mathrm{Aut}(M_n)$.

It is interesting that the inertia groups of $X_n \subset P(n+1) = (\mathbb{P}^1)^{n+1}$ and $M_n \subset \mathbb{P}^{n+1}$ have completely different structures though both X_n and M_n are Calabi-Yau hypersurfaces in rational Fano manifolds.

Remark 1.5. There is a smooth quartic $K3$ surface $M_2 \subset \mathbb{P}^3$ such that γ is not surjective (see [Og13, Theorem 1.2 (2)]). In particular, Theorem 1.4 is not true for $n = 2$.

2. Preliminaries

In this section, we prepare some definitions and properties of birational geometry and introduce the Cremona group.

2.1. **Divisors and singularities.** Let X be a projective variety. A *prime divisor* on X is an irreducible subvariety of codimension one, and a *divisor* (resp. $\mathbb{Q}$-*divisor* or $\mathbb{R}$-*divisor*) on X is a formal linear combination $D = \sum d_i D_i$ of prime divisors where $d_i \in \mathbb{Z}$ (resp. $\mathbb{Q}$ or $\mathbb{R}$). A divisor D is called *effective* if $d_i \geq 0$ for every i and denote $D \geq 0$. The closed set $\bigcup_i D_i$ of the union of prime divisors is called the *support* of D and denote $\mathrm{Supp}(D)$. A $\mathbb{Q}$-divisor D is called $\mathbb{Q}$-*Cartier* if, for some $0 \neq m \in \mathbb{Z}$, mD is a Cartier divisor (i.e. a divisor whose divisorial sheaf $\mathcal{O}_X(mD)$ is an invertible sheaf), and X is called $\mathbb{Q}$-*factorial* if every divisor is $\mathbb{Q}$-Cartier.

Note that, since the regular local ring is the unique factorization domain, every divisor automatically becomes the Cartier divisor on the smooth variety.

Let $f \colon X \dashrightarrow Y$ be a birational map between normal projective varieties, D a prime divisor, and U the domain of definition of f; that is, the maximal subset of X such that there exists a morphism $f \colon U \to Y$. Then $\mathrm{codim}(X \setminus U) \geq 2$ and $D \cap U \neq \emptyset$, the image $(f|_U)(D \cap U)$ is a locally closed subvariety of Y. If the closure of that image is a prime divisor of Y, we call it the *strict transform* of D (also called the *proper transform* or *birational transform*) and denote f_*D. We define $f_*D = 0$ if the codimension of the image $(f|_U)(D \cap U)$ is ≥ 2 in Y.

We can also define the strict transform f_*Z for subvariety Z of large codimension; if $Z \cap U \neq \emptyset$ and dimension of the image $(f|_U)(Z \cap U)$ is equal to $\dim Z$, then we define f_*Z as the closure of that image, otherwise $f_*Z = 0$.

Let (X, D) is a *log pair* which is a pair of a normal projective variety X and a $\mathbb{R}$-divisor $D \geq 0$. For a log pair (X, D), it is more natural to consider a *log canonical divisor* $K_X + D$ instead of a canonical divisor K_X.

A projective birational morphism $g \colon Y \to X$ is a *log resolution* of the pair (X, D) if Y is smooth, $\mathrm{Exc}(g)$ is a divisor, and $g_*^{-1}(D) \cup \mathrm{Exc}(g)$ has simple normal crossing support (i.e. each components is a smooth divisor and all components meet transversely) where $\mathrm{Exc}(g)$ is an exceptional set of g, and a divisor *over* X is a divisor E on some smooth variety Y endowed with a proper birational morphism $g \colon Y \to X$.

If we write

$$K_Y + \Gamma + \sum E_i = g^*(K_X + D) + \sum a_{E_i}(X, D) E_i,$$

where Γ is the strict transform of D and E_i runs through all prime exceptional divisors, then the numbers $a_{E_i}(X, D)$ is called the *discrepancies of* (X, D) *along* E_i. The *discrepancy of* (X, D) is given by

$$\operatorname{discrep}(X, D) := \inf\{a_{E_i}(X, D) \mid E_i \text{ is a prime exceptional divisor over } X\}.$$

The discrepancy $a_{E_i}(X, D)$ along E_i is independent of the choice of birational maps g and only depends on E_i.

Let us denote $\operatorname{discrep}(X, D) = a_E$. A pair (X, D) is *log canonical* (resp. *Kawamata log terminal* (*klt*)) if $a_E \geq 0$ (resp. $a_E > 0$). A pair (X, D) is *canonical* (resp. *terminal*) if $a_E \geq 1$ (resp. $a_E > 1$).

2.2. **Cremona groups.** Let n be a positive integer. The *Cremona group* $\mathrm{Cr}(n)$ is the group of automorphisms of $\mathbb{C}(X_1, \ldots, X_n)$, the $\mathbb{C}$-algebra of rational functions in n independent variables.

Given n rational functions $F_i \in \mathbb{C}(X_1, \ldots, X_n)$, $1 \leq i \leq n$, there is a unique endomorphism of this algebra maps X_i onto F_i and this is an automorphism if and only if the rational transformation f defined by $f(X_1, \ldots, X_n) = (F_1, \ldots, F_n)$ is a birational transformation of the affine space $\mathbb{A}^n$. Compactifying $\mathbb{A}^n$, we get

$$\mathrm{Cr}(n) = \mathrm{Bir}(\mathbb{A}^n) = \mathrm{Bir}(\mathbb{P}^n)$$

where $\mathrm{Bir}(X)$ denotes the group of all birational transformations of X.

In the end of this section, we define two subgroups in $\mathrm{Cr}(n)$ introduced by Gizatullin [Gi94].

Definition 2.1. Let V be an $(n+1)$-dimensional smooth projective rational manifold and $X \subset V$ a projective variety. The *decomposition group* of X is the group

$$\mathrm{Dec}(V, X) := \{f \in \mathrm{Bir}(V) \mid f(X) = X \text{ and } f|_X \in \mathrm{Bir}(X)\}.$$

The *inertia group* of X is the group

$$\mathrm{Ine}(V, X) := \{f \in \mathrm{Dec}(V, X) \mid f|_X = \mathrm{id}_X\}.$$

The decomposition group is also denoted by $\mathrm{Bir}(V, X)$. By the definition, the correspondence

$$\gamma\colon \mathrm{Dec}(V, X) \ni f \mapsto f|_X \in \mathrm{Bir}(X)$$

defines the exact sequence:

$$1 \longrightarrow \mathrm{Ine}(V, X) = \ker\gamma \longrightarrow \mathrm{Dec}(V, X) \xrightarrow{\gamma} \mathrm{Bir}(X). \tag{2.1}$$

So, it is natural to consider the following question (which is same as Question 1.2) asked by Ludmil Katzarkov:

Question 2.2. Is the sequence

$$1 \longrightarrow \mathrm{Ine}(V, X) \longrightarrow \mathrm{Dec}(V, X) \xrightarrow{\gamma} \mathrm{Bir}(X) \longrightarrow 1 \tag{2.2}$$

exact, i.e., is γ surjective?

Remark 2.3. In general, the above sequence (2.2) is not exact, i.e., γ is not surjective. In fact, there is a smooth quartic $K3$ surface $M_2 \subset \mathbb{P}^3$ such that γ is not surjective ([Og13, Theorem 1.2 (2)]).

3. Calabi-Yau hypersurface in $\mathbb{P}^{n+1}$

Our goal, in this section, is to prove Theorem 1.4 (i.e. Theorem 3.3). Before that, we introduce the result of Takahashi [Ta98].

Definition 3.1. Let X be a normal $\mathbb{Q}$-factorial projective variety. The 1-*cycle* is a formal linear combination $C = \sum a_i C_i$ of proper curves $C_i \subset X$ which are irreducible and reduced. By the theorem of the base of Néron-Severi (see [Kl66]), the whole numerical equivalent class of 1-cycle with real coefficients becomes the finite dimensional $\mathbb{R}$-vector space and denotes $N_1(X)$. The dimension of $N_1(X)$ or its dual $N^1(X)$ with respect to the intersection form is called the *Picard number* and denote $\rho(X)$.

Theorem 3.2. ([Ta98, Theorem 2.3]) *Let X be a Fano manifold (i.e. a manifold whose anti-canonical divisor $-K_X$ is ample,) with $\dim X \geq 3$ and $\rho(X) = 1$, $S \in |-K_X|$ a smooth hypersurface with $\mathrm{Pic}(X) \to \mathrm{Pic}(S)$ surjective. Let $\Phi\colon X \dashrightarrow X'$ be a birational map to a $\mathbb{Q}$-factorial terminal variety X' with $\rho(X') = 1$ which is not an isomorphism, and $S' = \Phi_* S$. Then $K_{X'} + S'$ is ample.*

This theorem is proved by using the *Noether-Fano inequality* which is one of the most important tools in birational geometry, which gives a precise bound on the singularities of indeterminacies of a birational map and some conditions when it becomes isomorphism.

This inequality is essentially due to [IM71], and Corti proved the general case of an arbitrary Mori fiber space of dimension three [Co95]. It was extended in all dimensions in [Ta95], [BM97], [Is01], and [dFe02], (see also [Ma02]). In particular, a log generalized version obtained independently in [BM97], [Ta95] is used for the proof of Theorem 3.2.

After that, we consider n-dimensional *Calabi-Yau manifold* X in this paper. It is a projective manifold which is simply connected,

$$H^0(X, \Omega_X^i) = 0 \quad (0 < i < \dim X = n), \quad \text{and} \quad H^0(X, \Omega_X^n) = \mathbb{C}\omega_X,$$

where ω_X is a nowhere vanishing holomorphic n-form.

The following theorem is a consequence of the Theorem 3.2, which is same as Theorem 1.4. This provides an example of the Calabi-Yau hypersurface M_n whose inertia group consists of only identity transformation.

Theorem 3.3. *Suppose $n \geq 3$. Let $M_n = (n+2) \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $n+2$. Then M_n is a Calabi-Yau manifold of dimension n and Question 2.2 is affirmative for M_n. More precisely:*

1. $\mathrm{Dec}(\mathbb{P}^{n+1}, M_n) = \{f \in \mathrm{PGL}(n+2, \mathbb{C}) = \mathrm{Aut}(\mathbb{P}^{n+1}) \mid f(M_n) = M_n\}$.
2. $\mathrm{Ine}(\mathbb{P}^{n+1}, M_n) = \{\mathrm{id}_{\mathbb{P}^{n+1}}\}$, *and* $\gamma\colon \mathrm{Dec}(\mathbb{P}^{n+1}, M_n) \xrightarrow{\simeq} \mathrm{Bir}(M_n) = \mathrm{Aut}(M_n)$.

Proof. By Lefschetz hyperplane section theorem for $n \geq 3$, $\pi_1(M_n) \simeq \pi_1(\mathbb{P}^{n+1}) = \{\mathrm{id}\}$, $\mathrm{Pic}(M_n) = \mathbb{Z}h$ where h is the hyperplane class. By the adjunction formula,

$$K_{M_n} = (K_{\mathbb{P}^{n+1}} + M_n)|_{M_n} = -(n+2)h + (n+2)h = 0$$

in $\mathrm{Pic}(M_n)$.

By the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^{n+1}}(-(n+2)) \longrightarrow \mathcal{O}_{\mathbb{P}^{n+1}} \longrightarrow \mathcal{O}_{M_n} \longrightarrow 0$$

and

$$h^k(\mathcal{O}_{\mathbb{P}^{n+1}}(-(n+2))) = 0 \ \text{ for } \ 1 \leq k \leq n,$$
$$H^k(\mathcal{O}_{M_n}) \simeq H^k(\mathcal{O}_{\mathbb{P}^{n+1}}) = 0 \ \text{ for } \ 1 \leq k \leq n-1.$$

Hence $H^0(\Omega^k_{M_n}) = 0$ for $1 \leq k \leq n-1$ by the Hodge symmetry. Hence M_n is a Calabi-Yau manifold of dimension n.

By $\mathrm{Pic}(M_n) = \mathbb{Z}h$, there is no small projective contraction of M_n, in particular, M_n has no flop. Thus by Kawamata [Ka08], we get $\mathrm{Bir}(M_n) = \mathrm{Aut}(M_n)$, and $g^*h = h$ for $g \in \mathrm{Aut}(M_n) = \mathrm{Bir}(M_n)$.

So we have $g = \tilde{g}|_{M_n}$ for some $\tilde{g} \in \mathrm{PGL}(n+1, \mathbb{C})$. Assume that $f \in \mathrm{Dec}(\mathbb{P}^{n+1}, M_n)$. Then $f_*(M_n) = M_n$ and $K_{\mathbb{P}^{n+1}} + M_n = 0$. Thus by Theorem 3.2, $f \in \mathrm{Aut}(\mathbb{P}^{n+1}) = \mathrm{PGL}(n+2, \mathbb{C})$. This proves (1) and the surjectivity of γ.

Let $f|_{M_n} = \mathrm{id}_{M_n}$ for $f \in \mathrm{Dec}(\mathbb{P}^{n+1}, M_n)$. Since $f \in \mathrm{PGL}(n+1, \mathbb{C})$ by (1) and M_n generates $\mathbb{P}^{n+1}$, i.e., the projective hull of M_n is $\mathbb{P}^{n+1}$, it follows that $f = \mathrm{id}_{\mathbb{P}^{n+1}}$ if $f|_{M_n} = \mathrm{id}_{M_n}$. Hence $\mathrm{Ine}(\mathbb{P}^{n+1}, M_n) = \{\mathrm{id}_{\mathbb{P}^{n+1}}\}$, i.e., γ is injective. So, $\gamma\colon \mathrm{Dec}(\mathbb{P}^{n+1}, M_n) \xrightarrow{\simeq} \mathrm{Bir}(M_n) = \mathrm{Aut}(M_n)$. □

4. Calabi-Yau hypersurface in $(\mathbb{P}^1)^{n+1}$

As in above section, the Calabi-Yau hypersurface M_n of $\mathbb{P}^{n+1}$ with $n \geq 3$ has only identical transformation as the element of its inertia group. However, there exist some Calabi-Yau hypersurfaces in the product of $\mathbb{P}^1$ which does not satisfy this property; as result (Theorem 4.2) shows.

To simplify, we denote

$$P(n+1) \coloneqq (\mathbb{P}^1)^{n+1} = \mathbb{P}^1_1 \times \mathbb{P}^1_2 \times \cdots \times \mathbb{P}^1_{n+1},$$
$$P(n+1)_i \coloneqq \mathbb{P}^1_1 \times \cdots \times \mathbb{P}^1_{i-1} \times \mathbb{P}^1_{i+1} \times \cdots \times \mathbb{P}^1_{n+1} \simeq P(n),$$

and

$$p^i\colon P(n+1) \to \mathbb{P}^1_i \simeq \mathbb{P}^1,$$
$$p_i\colon P(n+1) \to P(n+1)_i$$

as the natural projection. Let H_i be the divisor class of $(p^i)^*(\mathcal{O}_{\mathbb{P}^1}(1))$, then $P(n+1)$ is a Fano manifold of dimension $n+1$ and its canonical divisor has the form $-K_{P(n+1)} = \displaystyle\sum_{i=1}^{n+1} 2H_i$. Therefore, by the adjunction formula, the generic hypersurface $X_n \subset P(n+1)$ has trivial canonical divisor if and only if it has multidegree $(2, 2, \ldots, 2)$. More strongly, for $n \geq 3$, $X_n = (2, 2, \ldots, 2)$ becomes a Calabi-Yau manifold of dimension n and, for $n = 2$, a $K3$ surface (i.e. 2-dimensional Calabi-Yau manifold). This is shown by the same method as in the proof of Theorem 3.3.

From now, X_n is a generic hypersurface of $P(n+1)$ of multidegree $(2, 2, \ldots, 2)$ with $n \geq 3$. Let us write $P(n+1) = \mathbb{P}^1_i \times P(n+1)_i$. Let $[x_{i1} : x_{i2}]$ be the homogeneous coordinates of $\mathbb{P}^1_i$. Hereafter, we consider the affine locus and denote by $x_i = \dfrac{x_{i2}}{x_{i1}}$ the affine coordinates of $\mathbb{P}^1_i$ and by $\mathbf{z}_i$ that of $P(n+1)_i$. When we pay attention to x_i, X_n can be written by following equation

$$X_n = \{F_{i,0}(\mathbf{z}_i)x_i^2 + F_{i,1}(\mathbf{z}_i)x_i + F_{i,2}(\mathbf{z}_i) = 0\} \tag{4.1}$$

where each $F_{i,j}(\mathbf{z}_i)$ $(j = 0, 1, 2)$ is a quadratic polynomial of $\mathbf{z}_i$. Now, we consider the two involutions of $P(n+1)$:

$$\tau_i \colon (x_i, \mathbf{z}_i) \to \left(-x_i - \frac{F_{i,1}(\mathbf{z}_i)}{F_{i,0}(\mathbf{z}_i)}, \mathbf{z}_i\right) \tag{4.2}$$

$$\sigma_i \colon (x_i, \mathbf{z}_i) \to \left(\frac{F_{i,2}(\mathbf{z}_i)}{x_i \cdot F_{i,0}(\mathbf{z}_i)}, \mathbf{z}_i\right). \tag{4.3}$$

Then $\tau_i|_{X_n} = \sigma_i|_{X_n} = \iota_i$ by definition of ι_i (cf. Theorem 1.1).

We get two birational automorphisms of X_n

$$\rho_i = \sigma_i \circ \tau_i \colon (x_i, \mathbf{z}_i) \to \left(\frac{F_{i,2}(\mathbf{z}_i)}{-x_i \cdot F_{i,0}(\mathbf{z}_i) - F_{i,1}(\mathbf{z}_i)},\ \mathbf{z}_i\right)$$

$$\rho_i' = \tau_i \circ \sigma_i \colon (x_i, \mathbf{z}_i) \to \left(-\frac{x_i \cdot F_{i,1}(\mathbf{z}_i) + F_{i,2}(\mathbf{z}_i)}{x_i \cdot F_{i,0}(\mathbf{z}_i)},\ \mathbf{z}_i\right).$$

Obviously, both ρ_i and ρ_i' are in $\mathrm{Ine}(P(n+1), X_n)$, map points not in X_n to other points also not in X_n, and $\rho_i^{-1} = \rho_i'$ by $\tau_i^2 = \sigma_i^2 = \mathrm{id}_{P(n+1)}$.

Proposition 4.1. *Each ρ_i has infinite order.*

Proof. By the definiton of ρ_i and $\rho_i' = \rho_i^{-1}$, it suffices to show

$$\begin{pmatrix} 0 & F_{i,2} \\ -F_{i,0} & -F_{i,1} \end{pmatrix}^k \neq \alpha I$$

for any $k \in \mathbb{Z} \setminus \{0\}$ where I is an identity matrix and $\alpha \in \mathbb{C}^\times$. Their eigenvalues are

$$\frac{-F_{i,1} \pm \sqrt{F_{i,1}^2 - 4F_{i,0}F_{i,2}}}{2}.$$

Here $F_{i,1}^2 - 4F_{i,0}F_{i,2} \neq 0$ as X_n is general (for all i).

If $\begin{pmatrix} 0 & F_{i,2} \\ -F_{i,0} & -F_{i,1} \end{pmatrix}^k = \alpha I$ for some $k \in \mathbb{Z} \setminus \{0\}$ and $\alpha \in \mathbb{C}^\times$, then

$$\left(\frac{-F_{i,1} + \sqrt{F_{i,1}^2 - 4F_{i,0}F_{i,2}}}{-F_{i,1} - \sqrt{F_{i,1}^2 - 4F_{i,0}F_{i,2}}}\right)^k = 1,$$

a contradiction to the assumption that X_n is generic. $\square$

We also remark that Proposition 4.1 is also implicitly proved in Theorem 4.2.

Our main result is the following (which is same as Theorem 1.3):

Theorem 4.2. *Let $X_n \subset P(n+1)$ be a smooth hypersurface of multidegree $(2, 2, \ldots, 2)$ and $n \geq 3$. Then:*

(1) $\gamma \colon \mathrm{Dec}(P(n+1), X_n) \to \mathrm{Bir}(X_n)$ *is surjective, in particular Question* 2.2 *is affirmative for X_n.*

(2) *If, in addition, X_n is generic, $n+1$ elements $\rho_i \in \mathrm{Ine}(P(n+1), X_n)$ $(1 \leq i \leq n+1)$ satisfy*

$$\langle \rho_1, \rho_2, \ldots, \rho_{n+1} \rangle \simeq \underbrace{\mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}}_{n+1} \subset \mathrm{Ine}(P(n+1), X_n).$$

In particular, $\mathrm{Ine}(P(n+1), X_n)$ *is an infinite non-commutative group.*

Let $\mathrm{Ind}(\rho)$ be the union of the indeterminacy loci of each ρ_i and ρ_i^{-1}; that is, $\mathrm{Ind}(\rho) = \bigcup_{i=1}^{n+1} \left(\mathrm{Ind}(\rho_i) \cup \mathrm{Ind}(\rho_i^{-1})\right)$ where $\mathrm{Ind}(\rho_i)$ is the indeterminacy locus of ρ_i. Clearly, $\mathrm{Ind}(\rho)$ has codimension ≥ 2 in $P(n+1)$.

Proof. Let us show Theorem 4.2 (1). Suppose X_n is generic. For a general point $x \in P(n+1)_i$, the set $p_i^{-1}(x)$ consists of two points. When we put these two points y and y', then the correspondence $y \leftrightarrow y'$ defines a natural birational involutions of X_n, and this is the involution ι_j. Then, by Cantat-Oguiso's result [CO11, Theorem 3.3 (4)], $\mathrm{Bir}(X_n)$ $(n \geq 3)$ coincides with the group $\langle \iota_1, \iota_2, \ldots, \iota_{n+1} \rangle \simeq \underbrace{\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \cdots * \mathbb{Z}/2\mathbb{Z}}_{n+1}$.

Two involutions τ_j and σ_j of X_n which we construct in (4.2) and (4.3) are the extensions of the covering involutions ι_j. Hence, $\tau_j|_{X_n} = \sigma_j|_{X_n} = \iota_j$. Thus γ is surjective. Since automorphisms of X_n come from that of total space $P(n+1)$, it holds the case that X_n is not generic. This completes the proof of Theorem 4.2 (1).

Then, we show Theorem 4.2 (2). By Proposition 4.1, order of each ρ_i is infinite. Thus it is sufficient to show that there is no non-trivial relation between its $n+1$ elements ρ_i. We show by arguing by contradiction.

Suppose to the contrary that there is a non-trivial relation between $n+1$ elements ρ_i, that is, there exists some positive integer N such that

$$\rho_{i_1}^{n_1} \circ \rho_{i_2}^{n_2} \circ \cdots \circ \rho_{i_l}^{n_l} = \mathrm{id}_{P(n+1)} \tag{4.4}$$

where l is a positive integer, $n_k \in \mathbb{Z} \setminus \{0\}$ $(1 \leq k \leq l)$, and each ρ_{i_k} denotes one of the $n+1$ elements ρ_i $(1 \leq i \leq n+1)$ and satisfies $\rho_{i_k} \neq \rho_{i_{k+1}}$ $(0 \leq k \leq l-1)$. Put $N = |n_1| + \cdots + |n_l|$.

In the affine coordinates $(x_{i_1}, \mathbf{z}_{i_1})$ where x_{i_1} is the affine coordinates of i_1-th factor $\mathbb{P}^1_{i_1}$, we can choose two distinct points $(\alpha_1, \mathbf{z}_{i_1})$ and $(\alpha_2, \mathbf{z}_{i_1})$, $\alpha_1 \neq \alpha_2$, which are not included in both X_n and $\mathrm{Ind}(\rho)$.

By a suitable projective linear coordinate change of $\mathbb{P}^1_{i_1}$, we can set $\alpha_1 = 0$ and $\alpha_2 = \infty$. When we pay attention to the i_1-th element x_{i_1} of the new coordinates, we put same letters $F_{i_1,j}(\mathbf{z}_{i_1})$ for the definitional equation of X_n, that is, X_n can be written by

$$X_n = \{F_{i_1,0}(\mathbf{z}_{i_1})x_{i_1}^2 + F_{i_1,1}(\mathbf{z}_{i_1})x_{i_1} + F_{i_1,2}(\mathbf{z}_{i_1}) = 0\}.$$

Here the two points $(0, \mathbf{z}_{i_1})$ and $(\infty, \mathbf{z}_{i_1})$ not included in $X_n \cup \mathrm{Ind}(\rho)$. From the assumption, both two equalities hold:

$$\begin{cases} \rho_{i_1}^{n_1} \circ \cdots \circ \rho_{i_l}^{n_l}(0, \mathbf{z}_{i_1}) = (0, \mathbf{z}_{i_1}) & (4.5) \\ \rho_{i_1}^{n_1} \circ \cdots \circ \rho_{i_l}^{n_l}(\infty, \mathbf{z}_{i_1}) = (\infty, \mathbf{z}_{i_1}). & (4.6) \end{cases}$$

We proceed by dividing into the following two cases.

(i). The case where $n_1 > 0$. Write $\rho_{i_1} \circ \rho_{i_1}^{n_1-1} \circ \rho_{i_2}^{n_2} \circ \cdots \circ \rho_{i_l}^{n_l} = \mathrm{id}_{P(n+1)}$.

Let us denote $\rho_{i_1}^{n_1-1} \circ \cdots \circ \rho_{i_l}^{n_l}(0, \mathbf{z}_{i_1}) = (p, \mathbf{z}'_{i_1})$, then, by the definition of ρ_{i_1}, it maps p to 0. That is, the equation $F_{i_1,2}(\mathbf{z}'_{i_1}) = 0$ is satisfied. On the other hand, the intersection of X_n and the hyperplane $(x_{i_1} = 0)$ is written by

$$X_n \cap (x_{i_1} = 0) = \{F_{i_1,2}(\mathbf{z}_{i_1}) = 0\}.$$

This implies $(0, \mathbf{z}'_{i_1}) = \rho_{i_1}(p, \mathbf{z}'_{i_1}) = (0, \mathbf{z}_{i_1})$ is a point on X_n, a contradiction to the fact that $(0, \mathbf{z}_{i_1}) \notin X_n$.

(ii). The case where $n_1 < 0$. Write $\rho_{i_1}^{-1} \circ \rho_{i_1}^{n_1+1} \circ \rho_{i_2}^{n_2} \circ \cdots \circ \rho_{i_l}^{n_l} = \mathrm{id}_{P(n+1)}$.

By using the assumption (4.6), we lead the contradiction by the same way as in (i). Precisely, we argue as follows.

Let us write $x_{i_1} = \dfrac{1}{y_{i_1}}$, then $(x_{i_1} = \infty, \mathbf{z}_{i_1}) = (y_{i_1} = 0, \mathbf{z}_{i_1})$ and X_n and $\rho_{i_1}^{-1}$ can be written by

$$X_n \coloneqq \{F_{i_1,0}(\mathbf{z}_{i_1}) + F_{i_1,1}(\mathbf{z}_{i_1}) y_{i_1} + F_{i_1,2}(\mathbf{z}_{i_1}) y_{i_1}^2 = 0\},$$

$$\rho_{i_1}^{-1} \colon (y_{i_1}, \mathbf{z}_{i_1}) \to \left(-\frac{F_{i_1,0}(\mathbf{z}_{i_1})}{F_{i_1,1}(\mathbf{z}_{i_1}) + y_{i_1} \cdot F_{i_1,2}(\mathbf{z}_{i_1})},\ \mathbf{z}_{i_1} \right).$$

Let us denote $\rho_{i_1}^{n_1+1} \circ \rho_{i_2}^{n_2} \circ \cdots \circ \rho_{i_l}^{n_l}(y_{i_1} = 0, \mathbf{z}_{i_1}) = (y_{i_1} = q, \mathbf{z}''_{i_1})$, then $\rho_{i_1}^{-1}$ maps q to 0. That is, the equation $F_{i_1,0}(\mathbf{z}''_{i_1}) = 0$ is satisfied, but the intersection of X_n and the hyperplane $(y_{i_1} = 0)$ is written by

$$X_n \cap (y_{i_1} = 0) = \{F_{i_1,0}(\mathbf{z}_{i_1}) = 0\}.$$

This implies $(y_{i_1} = 0, \mathbf{z}''_{i_1}) = \rho_{i_1}^{-1}(y_{i_1} = q, \mathbf{z}''_{i_1}) = (x_{i_1} = \infty, \mathbf{z}_{i_1})$ is a point on X_n; that is, $(x_{i_1} = \infty, \mathbf{z}_{i_1}) \in X_n \cap (x_{i_1} = \infty)$. This is contradiction.

From (i) and (ii), we can conclude that there does not exist such N. This completes the proof of Theorem 4.2 (2). □

Note that, for the cases $n = 2$ and 1, Theorem 4.2 (2) also holds though (1) does not hold.

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References

[BM97] A. Bruno, K. Matsuki, *Log Sarkisov program*, Internat. J. Math. **8** no.4 (1997), 451-494.

[CO11] S. Cantat, K. Oguiso, *Birational automorphism group and the movable cone theorem for Calabi-Yau manifolds of Wehler type via universal Coxeter groups*, preprint arXiv:1107.5862, to appear in Amer. J. Math.

[Co95] A. Corti, *Factoring birational maps of threefolds after Sarkisov*, J. Algebraic Geom. **4** no.2 (1995), 223-254.

[dFe02] T. de Fernex, *Birational transformations of varieties*, University of Illinois at Chicago Ph. D. Thesis (2002).

[Gi94] M. H. Gizatullin, *The decomposition, inertia and ramification groups in birational geometry*, Algebraic Geometry and its Applications, Aspects of Math. **E25** (1994), 39-45.

[Is01] V. A. Iskovskikh, *Birational rigidity of Fano hypersurfaces in the framework of Mori theory*, Usp. Mat. Nauk **56** no.2 (2001), 3-86; English transl., Russ. Math. Surveys **56** no.2 (2001), 207-291.

[IM71] V. A. Iskovskikh, Yu. I. Manin, *Three-dimensional quartics and counterexamples to the Lüroth problem*, Mat. Sb. **86** no.1 (1971), 140-166; English transl., Math. Sb. **15** no.1 (1971), 141-166.

[Ka08] Y. Kawamata, *Flops connect minimal models*, Publ. Res. Inst. Math. Sci. **44** no.2 (2008), 419-423.

[Kl66] S. Kleiman, *Toward a numerical theory of ampleness*, Ann. of Math. **84** no.3 (1966), 293-344.

[Ma02] K. Matsuki, *Introduction to the Mori Program*, Universitext, Springer, New York (2002).

[Og13] K. Oguiso, *Quartic K3 surfaces and Cremona transformations*, Arithmetic and geometry of $K3$ surfaces and Calabi-Yau threefolds, Fields Inst. Commun. **67**, Springer, New York (2013), 455-460.

[Ta95] N. Takahashi, *Sarkisov program for log surfaces*, Tokyo University Master Thesis, 1995.

[Ta98] N. Takahashi, *An application of Noether-Fano inequalities*, Math. Z. **228** no.1 (1998), 1-9.

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